

Minimum Rectilinear Polygons for Given Angle Sequences

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Abstract

A *rectilinear* polygon is a polygon whose edges are axis-aligned. Walking counterclockwise on the boundary of such a polygon yields a sequence of left turns and right turns. The number of left turns always equals the number of right turns plus 4. It is known that any such sequence can be realized by a rectilinear polygon. In this paper, we consider the problem of finding realizations that minimize the perimeter or the area of the polygon or the area of the bounding box of the polygon. We show that all three problems are NP-hard in general. Then we consider the special cases of x -monotone and xy -monotone rectilinear polygons. For these, we can optimize the three objectives efficiently.

1 Introduction

In this paper, we consider the problem of computing, for a given rectilinear angle sequence, a “small” rectilinear polygon that realizes the sequence. A *rectilinear angle sequence* S is a sequence of left ($+90^\circ$) turns and right (-90°) turns, that is, $S = (s_1, \dots, s_n) \in \{\mathbb{L}, \mathbb{R}\}^n$, where n is the *length* of S . As we consider only rectilinear angle sequences, we usually drop the term “rectilinear.” A polygon P *realizes* an angle sequence S if there is a counterclockwise (*ccw*) walk along the boundary of P such that the turns at the vertices of P , encountered during the walk, form the sequence S . The turn at a vertex v of P is a left or right turn if the interior angle at v is 90° (v is convex) or, respectively, 270° (v is reflex).

In order to measure the size of a polygon, we only consider polygons that lie on the integer grid. Then, the *area* of a polygon P corresponds to the number of grid cells that lie in the interior of P . The *bounding box* of P is the smallest axis-parallel enclosing rectangle of P . The *perimeter* of P is the sum of the lengths of the edges of P . The task is, for a given angle sequence S , to find a polygon that realizes S and minimizes (i) (the area of) its bounding box, (ii) its area, or (iii) its perimeter. Figure 1 shows that, in general, the three criteria cannot be minimized simultaneously.

Obviously, the angle sequence of a polygon is unique (up to rotation), but the number of polygons that realize a given angle sequence is unbounded. The formula for the angle sum of a polygon implies that, in any angle sequence, $n = 2r + 4$, where r is the number of right turns, in other words, the number of right turns is exactly four less than the number of left turns.

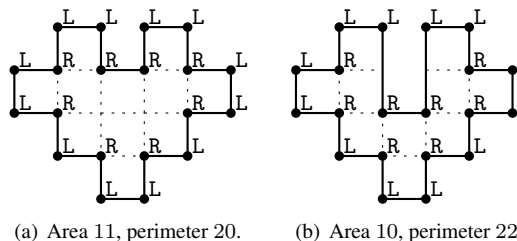


Figure 1: Two polygons realizing the same angle sequence. The bounding box of both polygons has area 20, but (a) has minimum perimeter and (b) has minimum area.

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Related Work. Bae et al. [1] considered, for a given angle sequence S , the polygon $P(S)$ that realizes S and minimizes its area. They studied the following question: Given a number n , find an angle sequence S of length n such that the area of $P(S)$ is minimized (and let $\delta(n)$ be this minimum area), or maximized (and let $\Delta(n)$ be this maximum area). They showed that (i) $\delta(n) = n/2 - 1$ if $n \equiv 4 \pmod 8$, $\delta(n) = n/2$ otherwise, and (ii) $\Delta(n) = (n-2)(n+4)/8$ for any $n \geq 4$. The result for $\Delta(n)$ tells us that any angle sequence S of length n can be realized by a polygon with area at most $(n-2)(n+4)/8$.

Several authors have explored the problem of realizing a turn sequence. Culberson and Rawlins [4] and Hartley [7] described algorithms that, given a sequence of exterior angles summing to 2π , construct a simple polygon realizing that angle sequence. Culberson and Rawlins’ algorithm, when constrained to $\pm 90^\circ$ angles, produces polygons with no colinear edges, implying that any n -vertex polygon can be drawn with area approximately $(n/2 - 1)^2$. However, as Bae et al. [1] showed, the bound is not tight.

In his PhD thesis, Sack [9] introduced label sequences (which are equivalent to turn sequences) and, among others, developed a grammar for label sequences that can be realized as simple rectilinear polygons.

Vijayan and Wigderson [11] considered the problem of efficiently embedding *rectilinear graphs*, of which rectilinear polygons are a special case, using an edge labeling that is equivalent to a turn sequence in the case of paths and cycles.

In graph drawing, the standard approach to drawing a graph of maximum degree 4 orthogonally (that is, with rectilinear edges) is the topology–shape–metrics approach of Tamassia [10]: (1) Compute a planar(ized) embedding; (2) compute an *orthogonal representation*, that is, an angle sequence for each edge and an angle for each vertex; (3) *compact* the graph, that is, draw it inside a bounding box of minimum area. Step (3) has been shown to be NP-complete by Patrignani [8]. Note that an orthogonal representation computed in step (2) is essentially an angle sequence for each face of the planarized embedding, so our problem corresponds to step (3) in the special case that the input graph is a simple cycle.

Another related work contains the reconstruction of a simple (non-rectilinear) polygon from partial geometric information. Disser et al. [5] constructed a simple polygon in $O(n^3 \log n)$ time from an ordered sequence of angles measured at the vertices visible from each vertex. The running time was improved to $O(n^2)$, which is the worst-case optimal [3]. Biedl et al. [2] considered polygon reconstruction from points (instead of angles) captured by laser scanning devices.

Our Contribution. First, we show that finding a minimum polygon that realizes a given angle sequence is NP-hard for any of the three measures: bounding box area, polygon area, and polygon perimeter; see Section 2. This extends the result of Patrignani [8] and settles an open question that he posed. We also give efficient algorithms for special types of angle sequences, namely *xy*- and *x-monotone sequences*, which are realized by *xy*-monotone and *x*-monotone polygons, respectively. (For example, LLRLLRLLRLLRLLRLLR is an *x*-monotone sequence, see Figure 1.) Our algorithms minimize area (Section 3) and perimeter (Section 4). For an overview of our results, see Table 1.

2 NP-Hardness of the General Case

In this section we show the NP-hardness of our problem for all three objectives: for minimizing the perimeter of the polygon, the area of the polygon, and the size of the bounding box. We first consider the following special problem from whose NP-hardness we then derive the three desired proofs.

Type of sequences	Minimum area	Min. bounding box	Minimum perimeter
general	NP-hard	NP-hard	NP-hard
<i>x</i> -monotone	$O(n^4)$	$O(n^3)$	$O(n^2)$
<i>xy</i> -monotone	$O(n)$	$O(n)$	$O(n)$

Table 1: Summary of our results.

FITUPPERRIGHT: Given an angle sequence S and positive integers W and H , is there a polygon realizing S within an axis-parallel rectangle R of width W and height H such that the first vertex of S lies in the upper right corner of R ?

Theorem 1. FITUPPERRIGHT is NP-hard.

Proof. Our proof is by reduction from 3-PARTITION: Given a multiset A of $n = 3m$ integers with $\sum_{a \in A} a = mB$, is there a partition of A into m subsets A_1, \dots, A_m such that $\sum_{a \in A_i} a = B$ for each i ? It is known that 3-PARTITION is NP-hard even if B is polynomially bounded in n and, for every $a \in A$, we have $B/4 < a < B/2$, which implies that each of the subsets A_1, \dots, A_m must contain exactly three elements [6].

For the idea of our reduction, see Figure 2. For an instance $A = \{a_1, \dots, a_{3m}\}$ of 3-PARTITION, we construct an LR-sequence S that can be drawn inside an $(W \times H)$ -box R if and only if A is a yes-instance. The sequence S consists of a *wall*, and for each number $a_i \in A$, a *snail*, which in turn consists of a *connector* and a *spiral*.

The wall is a box (LLLL) whose top right corner corresponds to the start of S . The connectors are attached to the left side of the wall by introducing two R-vertices. A connector is a thin x -monotone polygon going to the left that can change its y -position $m - 1$ times.

In detail, the LR-sequence S is defined as follows where $\rho = Bm^3$ is the number of windings of the spirals:

$$\begin{aligned} S &= \text{LL snail}_1 \text{snail}_2 \dots \text{snail}_{3m} \text{LL}, \\ \text{snail}_i &= \text{R}(\text{LRRL})^{m-1} \text{spiral}_i (\text{RLLR})^{m-1} \text{R}, \\ \text{spiral}_i &= (\text{LLLL})^\rho \text{ladder}_i (\text{RRRR})^{\rho-1} \text{RR}, \\ \text{ladder}_i &= (\text{RRL})^{(a_i-1) \cdot m^2}. \end{aligned}$$

We choose W and H such that the spirals have to be arranged in m columns of three spirals each. Note that for any order of the numbers in A , we can route the connectors in a planar way such that the triplets of spirals that we desire end up in the same column. Additionally, in each column there must be enough space for the at most $3m$ connectors that go from the wall to spirals further left; see Figure 2.

We set $W = 4\rho m + 16m^2 = \Theta(Bm^4)$ and $H = 12\rho + 2Bm^2 + 6m = \Theta(Bm^3)$. If all spirals are tightly wound, their bounding boxes need total area $(4\rho - 2) \cdot (2Bm^3 + 12\rho m) = \Theta(B^2m^7)$. The idea of our proof is to show that if a spiral is not tightly wound, we need too much space. The space that is not occupied by spirals is $O(Bm^5)$ in any drawing inside R .

It is clear that our construction is polynomial. By construction, there is a polygon realizing S that fits into R if A is a yes-instance of 3-PARTITION. It remains to show that if S fits into R , then A is a yes-instance of 3-PARTITION.

Fix any feasible drawing of S and a spiral spiral_i . Since the first vertex of S has to lie in the upper right corner of R , observe that the 5th L of spiral_i has to lie in the interior of the bounding box of the first four Ls of spiral_i . Inductively it follows that, for $5 \leq j \leq 4\rho$, the j th L of spiral_i lies in the interior of the bounding box of the last four Ls of spiral_i . Hence, the drawing of ladder_i lies in the bounding box of the last four Ls of the $(\text{LLLL})^\rho$ sequence of spiral_i . By repeating a similar argument for the R vertices, we can observe that every RR edge in spiral_i is lying opposite to a longer LL-edge such that the bounding box spanned by both edges is interiorly empty and completely contained in the polygon. Thus, we can move the RR edge towards the LL edge and assume that the bounding box has width 1. For the last $4\rho - 1$ RR edges in spiral_i , we call the bounding box an *arm*.

Hence, any drawing of a spiral consists of a drawing of the ladder and $4\rho - 1$ arms around it. We group the arms into four groups; top, bottom, left, right, depending to which side of the ladder they are lying. Recall that each arm is represented by a pair of LL and RR-edges. We order the arms in each group from the outside to the inside, that is, by the order of their LL edges in S , and define the *level* of an arm as its position in this ordering. We say that *level i is wound tightly* if the distance of all arms of level i to the arms of level $i + 1$ is 1.

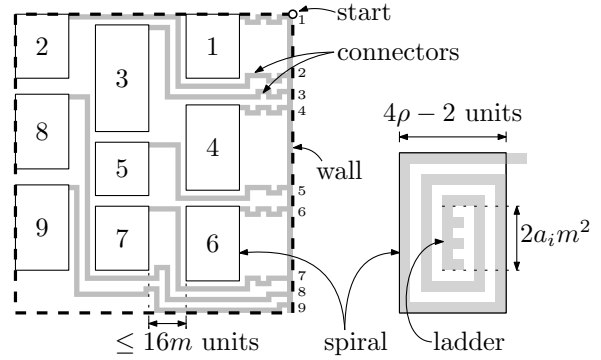


Figure 2: Spiral i has ρ windings; its height depends on the number a_i from the 3-PARTITION instance.

Observation 2. *If the first outer i levels are not wound tightly, then the spiral occupies $\Omega(i^2)$ more grid cells than in a tight winding.*

Proof. We consider only the length increase of the top arms. Since the spiral is not wound tightly, the horizontal distance between two consecutive left arms of the first outer i levels is at least two, one more than in a tightly wound spiral. The same is true for the right arms. Hence, the length of the level- i top arm increases at least by 2, that of the level- $(i-1)$ top arm at least by 4, and that of the level-1 top arm at least by $2i$; see Figure 3. Summing up the increases yields $\Omega(i^2)$. \triangle

Now, consider any feasible drawing. Recall that the space that is not occupied by spirals is $O(Bm^5)$. Hence, it follows by Observation 2 that at most the first $\lambda := O(\sqrt{B}m^{2.5})$ levels of any spiral are not wound tightly. We simplify the drawing by removing the wall, the connectors and the first λ levels of every spiral. We obtain a set of $3m$ disjoint rectangles, one for each snail. The rectangle for snail i is the bounding box of the inner $O(Bm^3 - \lambda) = O(Bm^3)$ levels of the snail's spiral, namely, those that must be wound tightly. Rectangle i has width $w := 4Bm^3 - 2 - 4\lambda$ and height $h_i := 4Bm^3 + 2a_i m^2 - 4\lambda$. Note that $h' := 4Bm^3 + Bm^2/2 - 4\lambda < h_i < 4Bm^3 + Bm^2 - 4\lambda$. If three rectangles share an x -coordinate, then the remaining height at this coordinate is at most

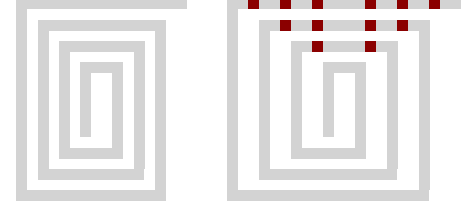


Figure 3: A spiral that is not wound tightly in the outer i levels occupies $\Omega(i^2)$ more area.

$$H - 3h' = 12Bm^3 + 2Bm^2 + 6m - 3(4Bm^3 + 1/2Bm^2 - 4\lambda) = Bm^2/2 + 6m - 4\lambda < h';$$

hence, no four rectangles can be drawn at a common x -coordinate. Further, if m rectangles share a y -coordinate, then the remaining width at this coordinate is $W - mw = 4Bm^4 + 16m^2 - m(4Bm^3 - 2 - 4\lambda) = 16m^2 - 2m - 4m\lambda < w$; hence, no $m+1$ rectangles can be drawn at a common y -coordinate.

These two facts combined imply an assignment of the rectangles to three rows of m rectangles each. To see this, consider three rectangles lying above each other. Then, since there is only $Bm^2/2 + 6m - 4\lambda < h'$ free vertical space, any rectangle has to be intersected by at least one of the three horizontal lines at y -coordinates $Bm^2/2 + 6m - 4\lambda + ih'$ with $i \in \{0, 1, 2\}$. No rectangle can intersect two lines, otherwise at most two rectangles would fit vertically and the third rectangle could not be squeezed in anywhere else. Analogously, we can assign the rectangles to one of the m columns by intersecting them with m vertical lines of distance w .

This assignment of rectangles to lines tells us the solution for the given instance of 3-PARTITION: for $i = 1, \dots, m$, we put into the set A_i the numbers $a_{i,1}, a_{i,2}, a_{i,3}$ represented by the three rectangles in column i . To complete our proof, we claim that $a_{i,1} + a_{i,2} + a_{i,3} \leq B$.

In order to see the claim, note that the λ removed levels of each spiral have to be wound completely around the corresponding rectangle. Thus, they also intersect the vertical line that goes through the rectangles in column i . Therefore, the height at this x -coordinate is at least $3 \cdot 4\rho + 2(a_{i,1} + a_{i,2} + a_{i,3})m^2$. The height and, hence, this expression is upperbounded by $H (= 12\rho + 2Bm^2 + 6m)$ since we assumed that the drawing fits into R . This yields $a_{i,1} + a_{i,2} + a_{i,3} \leq B + 3/m$. Exploiting that the $a_{i,j}$'s are integers shows that our above claim holds. \square

In order to show the NP-hardness of our three objectives, we adjust the above proof by attaching a very long spiral (with $\omega(WH)$, say $(WH)^2$, windings) to the wall such that it wraps around our construction above. Let T be the resulting LR-sequence. We will provide an upper bound for the objective value of T that holds if and only if the corresponding LR-sequence S is a yes-instance of 3-PARTITION. For this, we will use that any realization of S that is a no-instance causes the very long spiral to stretch by at least one unit horizontally or vertically, which makes the value of the objective increase above the mentioned upper bound.

In more detail, we construct the angle sequence T as follows (see Figure 4): We tightly draw a spiral around a rectangle of size $(W+1) \times (H+3)$ with $\omega(WH)$ windings. By adding the ladder $(\text{LLRR})^{W/2}$ to the innermost horizontal arm and the ladder $(\text{LLRR})^{H/2}$ to the innermost vertical arm of the spiral, we ensure that in any tight drawing with the two ladders being in the inside, the spiral goes around a rectangle of size exactly $W \times (H+2)$. Further, we add the ladder $(\text{LLRR})^{(4\omega(WH)+W)/2}$ to the outermost horizontal and the ladder $(\text{LLRR})^{(4\omega(WH)+H)/2}$ to the outermost vertical arm of the spiral. Finally, we add S to the spiral by using the appropriate one of the inner-most

arms of the spiral as the wall of S . Note that as long as S fits into a bounding box of size $W \times H$ it does not stretch the spiral around it. Hence, if and only if S is a yes-instance, we can draw S inside the spiral without stretching the spiral.

Consider any one of the two objectives: minimizing the inner area and minimizing the perimeter. Observe that in any drawing of S that fits inside the $(W \times H)$ -box, the value of the objective is bounded by $3WH$. Let t be the value of the objective of the spiral and its ladders when drawn tightly around a rectangle of size $W \times (H + 2)$. Then $t' := t + 3WH$ is an upper bound of the value of the objective of T in the case that S is a yes-instance.

Now assume that S is a no-instance. If the spiral is not winding around S , that is, if the bounding box of the first three arms of the spiral (starting with the arms with the attached $(\text{LLRR})^{WH}$ -ladders) does not contain S , then the other arms of the spiral have to be drawn outside the bounding box of the two arms. Hence, this increases the total length of the other arms by at least $\omega(WH)$, thus leading to a value of the objective greater than $t + \omega(WH) > t'$. If the spiral is winding around S , then, given that S is a no-instance, we have to stretch the spiral as argued above. Stretching the spiral by one unit in any direction, say in the horizontal direction, causes all $\omega(WH)$ many horizontal arms to increase by at least one unit. Hence, the value of the objective is at least $t + \omega(WH) > t'$.

The case of minimizing the bounding box is simpler: Let $W' \times H'$ be the size of the bounding box when the spiral and its ladders are drawn tightly around a rectangle of size $W \times (H + 2)$. We claim that T can be drawn inside an $(W' \times H')$ -bounding box if and only if S is a yes-instance. If S is not drawn inside the spiral, then the ladders $(\text{LLRR})^{WH}$ lie on the innermost arms of the spiral and the claim follows immediately. If S is drawn inside the spiral, we recall that S stretches the spiral (and thus the bounding box of T) if and only if it is a no-instance. This concludes the proof. \square

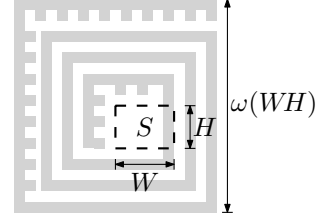


Figure 4: T containing S inside a long spiral.

3 The Monotone Case: Minimum Area

In this section, we show how to compute, for a monotone angle sequence, a polygon of minimum area. We start with the simple xy -monotone case and then consider the more general x -monotone case.

3.1 The xy -Monotone Case

An xy -monotone polygon has four *extreme edges*; its leftmost and rightmost vertical edge, and its topmost and bottommost horizontal edge. Two consecutive edges are connected by a (possible empty) xy -monotone chain that we will call a *stair*. Starting at the top extreme edge, we denote the four stairs in counterclockwise order TL , BL , BR , and TR ; see Figure 5(a). We say that an angle sequence consists of k nonempty *stair sequences* if any xy -monotone polygon that realizes it consists of k nonempty stairs; we also call it a k -*stair sequence*. The extreme edges correspond to the exactly four LL -sequences in an xy -monotone angle sequence and are unique up to rotation. Any xy -monotone angle sequence is of the form $[\text{L}(\text{LR})^*]^4$, where the single L describes the turn before an extreme edge and $(\text{LR})^*$ describes a stair sequence. W.l.o.g., we assume that an xy -monotone sequence always begins with LL and that we always draw the first LL as the topmost edge (the top extreme edge). Then, we can use TL , BL , BR , TR also for the corresponding stair sequences, namely the first, second third and forth $(\text{LR})^*$ subsequence after the first LL in cyclic order. Let T be the concatenation of TL , the top extreme edge, and TR ; let L , B , and R be defined analogously following Figure 5(a). For a chain C , let the R -length $r(C)$ be the number of reflex vertices on C .

Theorem 3. *Given an xy -monotone angle sequence S of length n , we can find a polygon P that realizes S and minimizes its (i) bounding box or (ii) area in $O(n)$ time, and in constant time we can find the optimum value of the objective if the R -lengths of the stair sequences are given.*

Proof. Part (i) of Theorem 3 follows from the following observation: The bounding box of every polygon that realizes S has width at least $\max\{r(T), r(B)\} + 1$ and height at least $\max\{r(L), r(R)\} + 1$. By drawing three stairs with edges of unit length, we can meet these lower bounds.

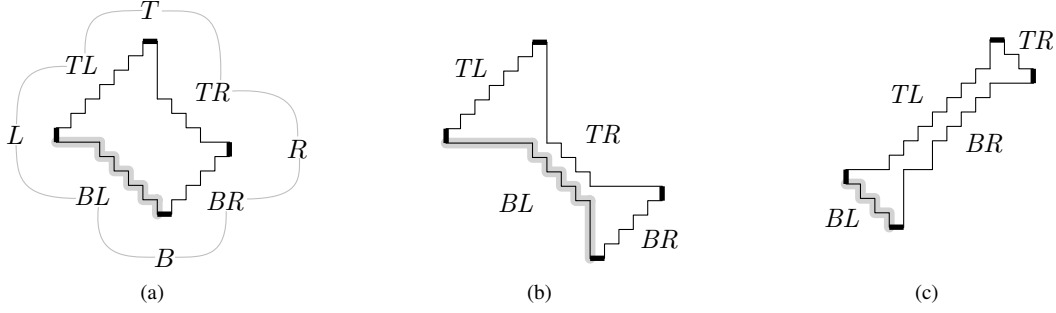


Figure 5: Extreme edges are bold. Stair BL is highlighted. (a) Notation: The four stairs TL , TR , BR , and BL of an xy -monotone polygon. The sequences T , R , B , and L are unions of neighboring stairs. (b) & (c) Two possible optimum configurations of the polygon.

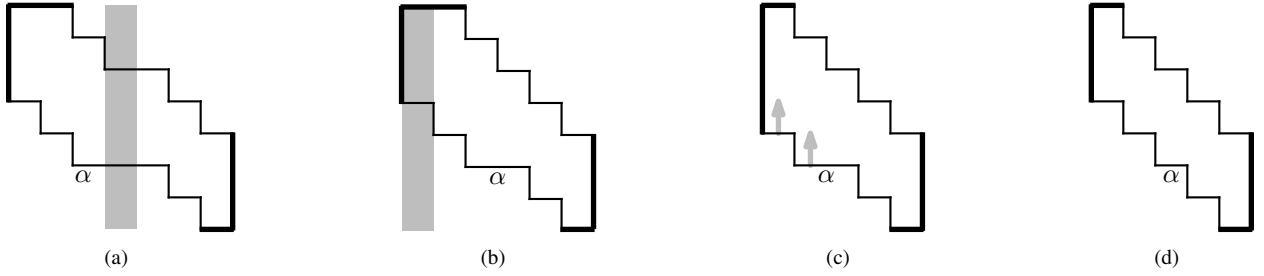


Figure 6: Reducing the horizontal segment α of BL to unit length while decreasing the total area. (a) If two long segments are above each other, contract both; (b) contract the leftmost long segment of TR ; (c) shift segments left of α and first part of α up; (d) resulting polygon.

For part (ii), we first consider angle sequences with at most two nonempty stairs. Here, the only non-trivial case is when the angle sequence consists of two opposite stair sequences, that is, TL and BR , or BL and TR . W.l.o.g., consider the second case.

Lemma 1. *Let S be an xy -monotone angle sequence of length n consisting of two nonempty opposite stair sequences BL and TR . We can find a minimum-area polygon that realizes S in $O(n)$ time. If $r(BL)$ and $r(TR)$ are given, we can compute the area of such a polygon in $O(1)$ time.*

Proof. Fix a minimum area polygon P that realizes S . Let $b := r(BL)$ and $a := r(TR)$. Assume (by rotation if necessary) that $b \geq a$. In the following, we consider the bottom and left extreme edge to be part of BL and the top and right extreme edge to be part of TR . Since P is of minimum area, we may assume that all horizontal segments of BL are of unit length. Otherwise, consider the leftmost horizontal segment α of BL longer than 1. If any horizontal segment of TR above it is longer than 1, then we may contract both by one unit and decrease the area of P without causing BL and TR to intersect; see Figure 6(a). If all such segments are of unit length then, since $b \geq a$, some horizontal segment of TR must be longer than 1 and have a unit-length horizontal segment of BL below it; see Figure 6(b). Take the leftmost pair and contract both by one unit, decreasing the area of P by at least 1 but removing one reflex vertex from BL . Add this reflex vertex back to BL by shifting the unit-length horizontal segments of BL between its last vertical segment of length at least 2 before α and the first unit-length piece of α up by one unit; see Figure 6(c). This also decreases the area of P and does not cause BL and TR to intersect. (Note: Such a vertical length segment must exist and no intersections are created because either TR consists of unit-length horizontal segments up to the x -coordinate of the right end of α or such a segment has been created in the contraction step.)

Let τ_i denote the i -th horizontal segment in TR (including the top extreme edge). The length $|\tau_i|$ of τ_i is also the number of horizontal BL -segments (including the bottom extreme edge) lying under τ_i . We have $\sum |\tau_i| = b + 1$.

Let $\text{area}(i)$ denote the area under τ_i in P . Since the left extreme edge in P has length at least 1, the area in P under τ_1 is $\text{area}(1) \geq \sum_{j=1}^{|\tau_1|} j = |\tau_1|(|\tau_1| + 1)/2$. For $i \geq 2$, $\text{area}(i) \geq \sum_{j=1}^{|\tau_i|} (j + 1) = (|\tau_i| + 1)(|\tau_i| + 2)/2 - 1$. We can overcome the difference between $i = 1$ and $i \geq 2$, by splitting τ_1 into τ'_0 and τ'_1 , such that $|\tau'_0| = 1$ and $|\tau'_1| = |\tau_1| - 1$. Let $\tau'_i = \tau_i$ for all other i . Observe that now $\sum_{i \geq 1} |\tau'_i| = b$. Thus,

$$\begin{aligned} \text{area}(P) &\geq 1 + \sum_{i \geq 1} \left(\frac{(|\tau'_i| + 1)(|\tau'_i| + 2)}{2} - 1 \right) \\ &= 1 + \sum_{i \geq 1} \left(\frac{1}{2} |\tau'_i|^2 + \frac{3}{2} |\tau'_i| \right) = 1 + \frac{3}{2} b + \frac{1}{2} \sum_{i \geq 1} |\tau'_i|^2, \end{aligned}$$

which is minimized if $\sum_{i \geq 1} |\tau'_i|^2$ is minimal. By Cauchy-Schwarz we know that this is the case if all $\tau'_{i \geq 1}$ are equal to the arithmetic mean; since we have to use integers, the convexity of the function tells us that all $\tau'_{i \geq 1}$ have to be as close to the arithmetic mean as possible, that is, $\tau'_{i \geq 1} \in \{\lfloor b/(a+1) \rfloor, \lceil b/(a+1) \rceil\}$. Hence,

$$\text{area}(P) \geq \frac{(a+1)(q+1)(q+2)}{2} - a + r(q+2)$$

where q is the quotient and r is the remainder when b is divided by $a+1$. This lower bound can be achieved provided $b > a$. If $b = a$, one can achieve only $2(b+1)$, which is 1 more than the bound, since the left extreme edge has length 2 (not 1) if all horizontal edges are unit length. \triangle

The proof of Lemma 1 leads to the following observation.

Observation 4. *In any polygon P of minimum area consisting of two nonempty opposite stairs BL and TR with $b := r(BL) \geq a := r(TR)$, BL consists of only unit-length segments and TR only of segments of lengths $\lfloor b/(a+1) \rfloor$ and $\lceil b/(a+1) \rceil$ (in any order).*

We now consider the case of four nonempty stairs. (The case of three nonempty stairs can be solved analogously.) An xy -monotone polygon P with four nonempty stairs TL , TR , BL , and BR is *canonical* if (C1) P has two non-adjacent nonempty stairs, say TL and BR , such that the bounding box B_{TL} of TL and its adjacent extreme edges and the bounding box B_{BR} of BR and its adjacent extreme edges intersect in at most one point, and (C2) the bottom-right corner of B_{TL} as well as the top-left corner of B_{BR} coincides with an endpoint of TR or BL .

Lemma 2. *For every 4-stair sequence S with $|S| > 36$, there exists a polygon of minimum area realizing S that is canonical.*

Proof. Consider an optimum polygon realizing angle sequence S . Assume it is not canonical. Observe that all four extreme edges are of length 1, otherwise the polygon is not optimum.

First, suppose that the canonical property (C1) does not hold. Then for any pair of two opposite stairs, the bounding boxes of their adjacent extreme edges intersect in more than one point. Hence, the (closed) x -ranges of the horizontal extreme edges intersect and the (closed) y -ranges of the vertical extreme edges intersect. Since the extreme edges have length 1, and the bounding boxes intersect in more than one point, we even have that either the (closed) x -ranges of the top and bottom extreme edges are the same, or the (closed) y -ranges of the left and right extreme edges are the same. Suppose (by rotation if necessary) it is the latter and also suppose (by temporary vertical or horizontal reflection and reflecting it back afterwards) that stair TR has R -length greater than 4 (since $|S| > 36$ this is possible). Let u be the left endpoint of the bottom extreme edge and let v be the reflex vertex that precedes (in ccw order) the top extreme edge; see Figure 7(a).

We shift the boundary of P that lies on the ccw walk from u to v down by two units, stretching the vertical edges adjacent to u and v . The new polygon P' still realizes the angle sequence and its area is larger by two units than the area of P . However, now B_{TL} and B_{BR} are intersection-free. Let w be the reflex vertex that follows (in ccw order) the right extreme edge and let z be the bottom endpoint of the left extreme edge; see Figure 7(b). We shift the boundary of P' that lies on the ccw walk from w to z to the left by three units, stretching the horizontal edges adjacent to w and z . The new polygon still realizes the angle sequence and is still simple: The only crossings that can occur by

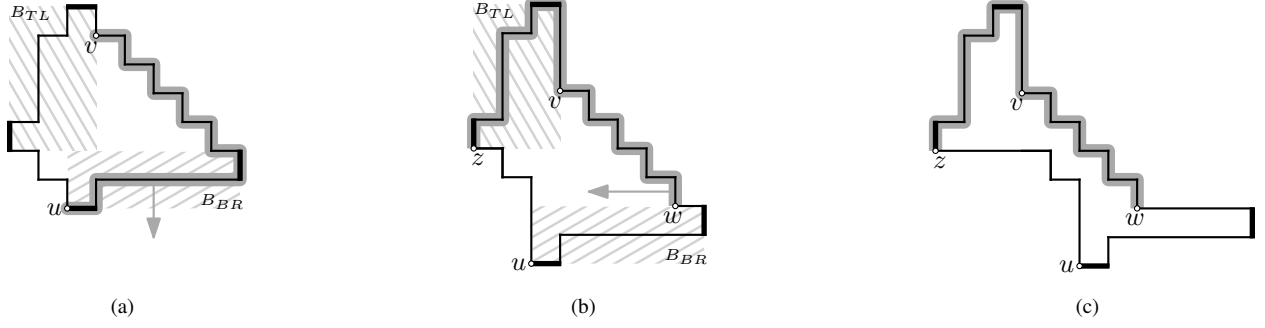


Figure 7: Transforming an xy -monotone polygon to a polygon that satisfies (C1) and has less area. (a) Shifting the ccw path between u and v down by two units; (b) shifting the ccw path between w and z left by three units; (c) the resulting polygon.

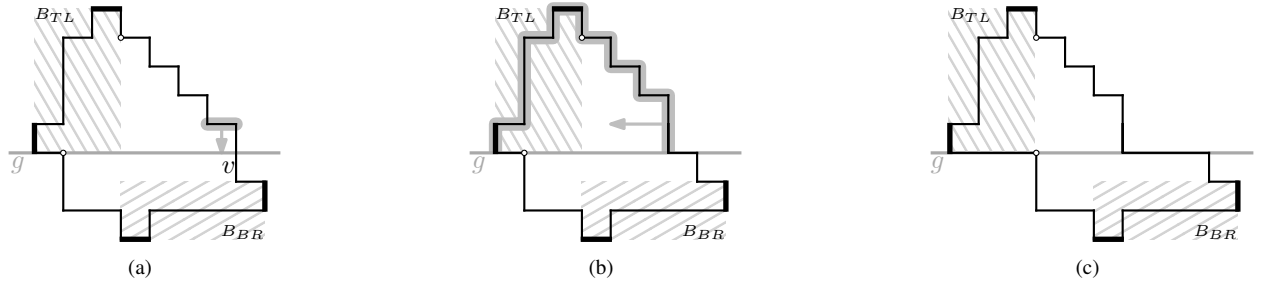


Figure 8: Transforming an xy -monotone polygon to a polygon that satisfies (C2) and has the same area. (a) Shifting a horizontal edge onto g ; (b) shifting the part above g to the left; (c) the resulting polygon.

this operation are between TR and BL . The left extreme edge lies at most three rows above the right extreme edge ρ ; hence, any crossing must involve the vertical edge e_1 of TR in the row above ρ or the vertical edge e_2 of TR two rows above ρ . Since $r(TR) > 4$, we have that (after the shift)

$$x(e_2) \geq x(e_1) \geq x(v) + r(TR) - 2 \geq x(v) + 3 = x(u) + 1.$$

Since each vertical edge of BL has x -coordinate at most $x(u)$, there can be no crossing; see Figure 7(c). However, now the area of the polygon decreased by three units; a contradiction to the fact that p is optimum. Hence, the canonical property (C1) has to hold.

Now, assume that there is a bounding box pair having at most one point in common, w.l.o.g. B_{TL} and B_{BR} . Since the optimum polygon is not canonical, the canonical property (C2) has to be violated. Hence, for at least one of the two bounding boxes, say B_{TL} , neither an endpoint of TR nor an endpoint of BL lies on a corner of B_{TL} , that is, their endpoints lie on two different edges of B_{TL} , and the distance from their endpoints to the closest corner of B_{TL} is at least 1. Then, for at least one of the two edges, it holds that the line going through the edge does not cross the interior of B_{BR} . W.l.o.g., this holds for the line g that goes through the horizontal edge of B_{TL} . Then, we can also observe that g does not cross any vertical line segment of TR ; instead, there is a horizontal line segment of TR lying on g .

To see this, assume the contrary. Then, there exists a vertical line segment v of TR that is cut by g ; see Figure 8(a). Thus, the two endpoints of v lie at least one unit above and below g , respectively. Consider the horizontal line segment of TR starting at the top endpoint of v . We can move the horizontal segment downwards and place it on g . By this, the angle sequence does not change and the polygon remains simple as all line segments of BL , the only segments that might cross TR after his operation, lie below g by at least one unit. Hence, by moving the horizontal edge downwards, we in fact shrink the polygon; a contradiction to its optimality. Thus, g contains a horizontal line segment of TR . Now, we cut the polygon through g into two parts; see Figure 8(b). Then, we shift the upper part to the left until BL intersects the bottom right corner of B_{TL} . The resulting polygon realizes the same angle sequence as before and

has the same area as before; see Figure 8(c). However, now BL intersects a corner of B_{TL} . If the polygon is not yet canonical, then we repeat the procedure with B_{BR} and get a canonical optimum polygon. Hence, the canonical property (C2) holds. \triangle

Consider the line segment of TR and the line segment of BL that connect to B_{TL} in a canonical polygon. The two line segments are (a) both horizontal, (b) both vertical, or (c) perpendicular to each other. Consequently, there is only a constant number of ways in which the stairs outside the two bounding boxes are connected to them. (The number of combinations is further limited as both case (a) and case (b) can appear only once.)

Consider a (canonical) optimum polygon. We cut the polygon along the edge of B_{TL} to which BL and TR are connected. We also cut along the respective edge of B_{BR} . We get three polygons. The polygons on the outside realize the 1-stair sequence defined by TL and BR (including the extreme edges), respectively, whereas the middle polygon realizes the 2-stair sequence defined by the concatenation of BL , TR , and the edge segments of B_{TL} and B_{BR} that connect them.

This observation leads to the following algorithm: For $|S| \leq 36$, we find a solution in constant time by exhaustive search. For larger $|S|$, we guess the partition of the extreme edges whose bounding boxes do not intersect in the (canonical) optimum polygon that we want to compute. W.l.o.g., we guessed B_{TL} and B_{BR} (the other case is symmetric). Then, we guess how TR and BL , the two stairs outside B_{TL} and B_{BR} , are connected to each of the two bounding boxes (see (a)–(c)). This gives us two 1-stair instances and a 2-stair instance. We solve the instances independently and then put the solutions together to form a solution to the whole instance. By Lemma 1 and Observation 4, we solve the middle instance such that the left extreme edge of our solution is of minimum length, and, if possible, also the top extreme edge.

In detail, we put them together as follows. Let P_1 denote our solution to the instance corresponding to B_{TL} and let P_2 denote our solution to the middle instance; see Figure 9(a). If we guessed case (a) for B_{TL} , then we put P_1 and P_2 together along their corresponding vertical extreme edges. If the right extreme edge of P_1 is too short, we make it sufficiently longer by lifting the top extreme edge of P_1 up. Case (b) works symmetrically. If we guessed (c) for B_{TL} , note that either the left or top extreme edge of P_2 has length at least 2. We put P_1 and P_2 along this extreme edge and the corresponding extreme edge of P_1 ; see Figure 9(b).

We repeat the same process with P_2 and our solution P_3 to the instance corresponding to B_{BR} . However, we proceed differently if the following holds: (a) we guessed case (a) or (b) for B_{BR} and the respective extreme edge of P_3 is too short for the corresponding edge e of P_2 , (b) we guessed (c) for B_{TL} or the respective extreme edge of P_1 is longer by at least 2 than the corresponding extreme edge f of P_2 , and (c) $\text{length}(f) < \text{length}(e)$. In this case, by Observation 4, we solve P_2 again such that e is of minimum length. Then we proceed as before. Note that P_1 and P_2 remain feasibly connected.

Let $b := r(BL) > a := r(TR)$. According to Observation 4, in P_2 , all horizontal segments in BL are of unit length and all horizontal segments in TR are of length $\lfloor b/(a+1) \rfloor$ or $\lceil b/(a+1) \rceil$.

All in all, we get a canonical polygon which realizes the given angle sequence. It follows immediately that the polygon has minimum area if we did not prolong any extreme edges in cases (a) or (b). Now, assume that we had to prolong an extreme edge of P_1 . W.l.o.g., we prolonged the bottom extreme edge of P_1 in case (b). Instead of prolonging the edge, we could have cut the polygon horizontally through the top endpoint of the left extreme edge (instead of through the bottom endpoint) and solve the two resulting instances, a 1- and a 2-stair instance, independently; see Figure 9(c). Observe that if we cut our combined solution in the same way, we get optimum solutions to those two instances. Let us consider our solution P'_2 to the 2-stair instance. We increased a (minimum-length) top step of TR by one and at the same time increased the number b of reflex vertices of BL by one. For BL (in P'_2), all steps are still of unit length, and for TR (in P'_2), only steps of lengths $\lfloor (b+1)/(a+1) \rfloor$ and $\lceil (b+1)/(a+1) \rceil$ appear.

Now, assume that we (also) prolonged an extreme edge of P_3 . W.l.o.g., we are in case (a). Consider the situation after a possible recomputation of P_2 . If the rightmost horizontal edge g on the top side of P_2 is of minimum length, then we can apply the same argument as before. Otherwise, we increase the length of any horizontal minimum-length edge on the top side of P_2 by one and reduce the length of g by one; by Observation 4, this remains a minimum-length solution.

Thus, we computed a polygon of minimum area. The running time is linear in n since our algorithm computes only constantly many 1-stair and 2-stair instances which are themselves solvable in linear time. Given the number of

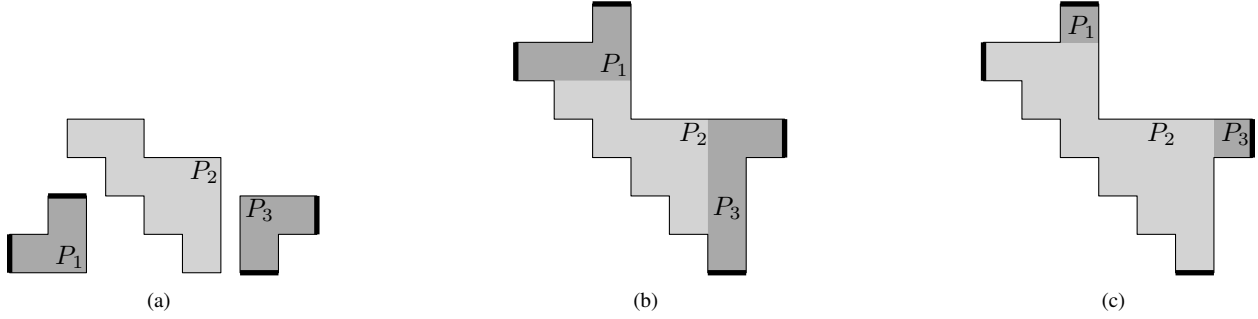


Figure 9: Putting the three solutions P_1 , P_2 and P_3 together. (a) The three solutions; (b) we use case (b) for P_1 and (a) for P_3 , both have to be stretched; (c) the alternative cut of P_1 and P_3 yields three new optimally solved instances.

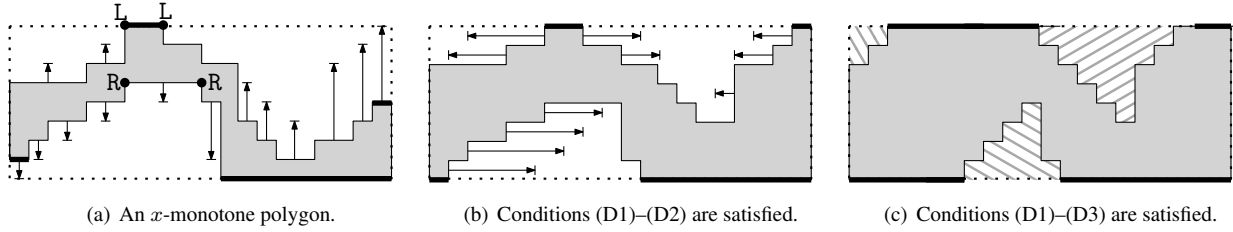


Figure 10: Illustration of how to make a polygon canonical. The thick horizontal edges are outer extreme edges, the tiling patterns mark double stairs (see definition in proof of Theorem 5). Note that the illustrating drawing is not optimal.

steps for the four stairs, we can even compute the minimum area in constant time since this is true for instances with two or less stairs. \square

3.2 The x -Monotone Case

For the x -monotone case, we first give an algorithm that minimizes the bounding box of the polygon, and then an algorithm that minimizes the area.

An x -monotone polygon consists of two *vertical extreme* edges, i.e., the leftmost and the rightmost vertical edge, and at least two *horizontal extreme* edges, which are defined to be the horizontal edges of locally maximum or minimum height. The vertical extreme edges divide the polygon into an upper and a lower hull, each of which consists of xy -monotone chains that are connected by the horizontal extreme edges. We call a horizontal extreme edge of type RR an *inner extreme edge*, and a horizontal extreme edge of type LL an *outer extreme edge*; see Figure 10(a). Similar to the xy -monotone case, we consider a *stair* to be an xy -monotone chain between any two consecutive extreme edges (outer and inner extreme edges as well as vertical extreme edges) and we denote by *stair sequence* the corresponding angle subsequence $(LR)^*$. W.l.o.g., at least one inner extreme edge exists, otherwise the polygon is xy -monotone and we refer to Section 3.1. Given an x -monotone sequence, we always draw the first RR -subsequence as the leftmost inner extreme edge of the lower hull. By this, the correspondence between the angle subsequences and the stairs and extreme edges is unique.

An x -monotone polygon is *canonical* if (D1) all outer extreme edges are lying on the border of the bounding box, (D2) each vertical non-extreme edge that is not incident to an inner extreme edge has length 1, and (D3) each horizontal edge that is not an outer extreme edge has length 1.

The following lemma states that it suffices to find a canonical x -monotone polygon of minimum bounding box; see Figure 10 for an illustration.

Lemma 3. *Any x -monotone polygon can be transformed into a canonical x -monotone polygon without increasing the area of its bounding box.*

Proof. Let P be an x -monotone polygon. We transform it into a canonical polygon in two steps.

First, we move all horizontal edges on the upper hull as far up as possible and all horizontal edges on the lower hull as far down as possible; see Figures 10(a)–(b). This establishes condition (D1). Furthermore, assume that there is a vertical edge (u, v) on the upper hull with $y(u) > y(v) + 1$. If the (unique) horizontal edge (v, w) is not an inner extreme edge, then it can be moved upwards until $y(u) = y(v) + 1$, which contradicts the assumption that all horizontal edges on the upper hull are moved as far up as possible. This argument applies symmetrically to the edges on the lower hull. Hence, condition (D2) is established.

Second, we move all vertical edges on a stair as far as possible in the direction of the inner extreme edge bounding the stair, e.g., if the stair lies on the upper hull and is directed downwards, then all vertical edges are moved as far right as possible; see Figures 10(b)–(c). This stretches the outer extreme edges while simultaneously contracting all other horizontal edges to length 1, which satisfies condition (D3).

Note that in neither step the bounding box changed. Since all conditions are satisfied, the resulting polygon is canonical. \square

We observe that the length of the vertical extreme edges depends on the height of the bounding box, while the length of all other vertical edges is fixed by the angle sequence. Thus, a canonical x -monotone polygon is fully described by the height of its bounding box and the length of its outer extreme edges. Furthermore, the y -coordinate of each vertex depends solely on the height of the bounding box.

We use a dynamic program that constructs a canonical polygon of minimum bounding box in time $O(n^3)$. For each possible height h of the bounding box, the dynamic program populates a table that contains an entry for any pair of an extreme vertex p (that is, an endpoint of an outer extreme edge) and a horizontal edge e of the opposite hull. The value of the entry $T[p, e]$ is the minimum width w such that the part of the polygon left of p can be drawn in a bounding box of height h and width w in such a way that the edge e is intersecting the interior of the grid column left of p .

Theorem 5. *Given an x -monotone angle sequence S of length n , we can find a polygon P that realizes S and minimizes the area of its bounding box in $O(n^3)$ time.*

Proof. To prove the theorem, we present an algorithm that constructs a canonical polygon of minimum bounding box in time $O(n^3)$. The height of any minimum bounding box is at most n ; otherwise, as there are only n vertices, there is a y -coordinate on the grid that contains no vertex and can be “removed”. For any height h of the n possible heights of an optimum polygon, we run the following dynamic program in $O(n^2)$ time.

We call the left and right endpoint of an outer extreme edge the *left extreme vertex* and the *right extreme vertex*, respectively. The dynamic program contains an entry for any pair of an extreme vertex p and a horizontal edge e of the opposite hull. Consider the part of the polygon between p and e that includes the left vertical extreme edge, that is, the chain that goes from p to e over the left vertical extreme edge. The value of the entry $T[p, e]$ is the minimum width w of a bounding box of height h in which this part of the polygon can be drawn in such a way that edge e is intersecting the interior of the grid column left of p and such that e has the same y -coordinate as it has in a canonical drawing of the whole polygon in a bounding box of height h ; see Figure 11. We call (p, e) an *extreme column pair*.

We compute $T[p, e]$ as follows. Consider a drawing of the part of the polygon between p and e that includes the left vertical extreme edge in a bounding box of height h and minimum width. Let p' be the rightmost extreme vertex in this drawing to the left of p , let (p', e') be the corresponding extreme column pair, and let w' be the horizontal distance between p and p' ; see Figure 11.

We can find (p', e') and w' from the angle sequence as follows. If p is a left extreme vertex, then, by condition (D3), the pair (p', e') and the distance w' is fully determined. Otherwise, if p is a right extreme vertex, then p' is either the left extreme vertex incident to p , or p' is the horizontally closest extreme vertex on the opposite hull; we test both cases. Again, by condition (D3), edge e' and distance w' is fully determined.

When determining (p', e') and w' , we also test, as we will describe in the next paragraph, whether we can canonically draw the part of the polygon between (p', e') and (p, e) in the given space constraints. If we can, then we call (p', e') a feasible pair for (p, e) . We find a feasible pair (p', e') for (p, e) with the smallest value of $T[p', e'] + w'$

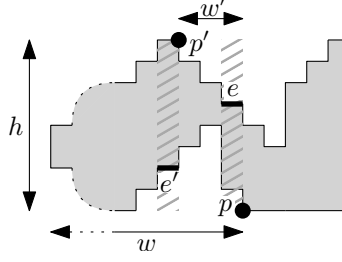


Figure 11: Two extreme column pairs (p, e) and (p', e') with $T[p, e] = T[p', e'] + w' = w$. The part of the polygon left of p can be drawn in the bounding box of size $h \times w$.

and set

$$T[p, e] = T[p', e'] + w'.$$

If all pairs for (p, e) are infeasible, we set $T[p, e] = \infty$.

First, we will argue that if there is such a canonical drawing, then it is unique. We assume that $T[p', e'] < \infty$. We group each pair of stairs that share an inner extreme edge as a *double stair*; see Figure 10(c). Each remaining stair forms a double stair by itself. Let P_{\top} denote the part of the upper hull between (p', e') and (p, e) . Given the choice of p' , it does not contain any endpoint of an outer extreme edge in its interior. Hence, there are only two cases. Either P_{\top} consists of a single horizontal line segment belonging to an outer extreme edge, or it is a subchain belonging to a double stair. In the first case, by condition (D1), we have to draw P_{\top} on the top boundary of the bounding box. Further, its left endpoint has x -coordinate equal to $T[p', e']$ and the length of the segment is w' . Hence, the drawing is unique. In the second case, note that conditions (D1)–(D3) determine the lengths and y -positions of all edges with exception of the lengths of the outer extreme edges. Thus, given the x -position of any vertex of a double stair, there is only one canonical way to draw the double stair. In our case, the value of $T[p', e']$ is equal to the x -position of the leftmost vertex of P_{\top} . Hence, the drawing of P_{\top} is unique. By the same arguments, we have to draw the part P_{\perp} of the lower hull between (p', e') and (p, e) in a unique way.

Now, given the unique drawings of P_{\top} and P_{\perp} , we check for every x -coordinate whether P_{\top} is lying above P_{\perp} . If and only if this is the case, then the two drawings together form a feasible canonical drawing and (p', e') is a feasible pair for (p, e) .

In the last step, we compute the minimum width w of the bounding box assuming height h . Consider an optimum canonical drawing of the whole polygon in a bounding box of height h . Let p^* be a rightmost (right) extreme vertex. Note that for p^* there are only two candidates, one from the upper hull and one from the lower hull. Since p^* is a rightmost extreme vertex, all horizontal edges to the right of p^* (on the upper and on the lower hull) are segments of length 1. Thus, given p^* , we can compute the distance r^* between p^* and the right vertical extreme edge. Let e^* be the r^* th horizontal edge from the right on the hull opposite to p^* . Observe that edge e^* is the edge that forms an extreme column pair with p^* . Hence, the width of the polygon is $w = T[p^*, e^*] + r^*$.

We compute width w as follows. For each one of the two candidates for p^* , we determine r^* and e^* . Then we check whether the candidate is feasible. For this, recall that conditions (D1)–(D3) determine the y -positions of all edges. Also recall that all horizontal edges to the right of (p^*, e^*) are of length 1. Hence, there is only one way to canonically draw the edges right to (p^*, e^*) . If the upper hull always stays above the lower hull, candidate p^* is feasible. Thus, we get the width by

$$w = \min_{\text{feasible candidate } p^*} \{T[p^*, e^*] + r^*\} \cup \{\infty\}.$$

For every height h , we compute the minimum width w and find the bounding box of minimum area $w \cdot h$.

It remains to show the running time of the algorithm. The table T consists of $O(n^2)$ entries. To find the value of an entry $T[p, e]$, we have to find the closest column pair (p', e') to the left, the distance w' , and we have to test whether we can canonically draw the polygon between (p', e') and (p, e) . We now show that each of these steps is possible in $O(1)$ time by precomputing some values for each point.

- (i) For each point, we store its y -coordinate. As observed above, the y -coordinate is fixed, and it can be computed in $O(n)$ time in total by traversing the stairs.
- (ii) For each point p , we store the next extreme point $\lambda(p)$ to the left on the same hull, as well as the distance $\delta(p)$ to it. These can be computed in $O(n)$ time by traversing the upper and the lower hull from left to right.
- (iii) For each left extreme vertex q , we store an array that contains all horizontal edges between q and $\lambda(q)$ ordered by their appearance on a walk from q to $\lambda(q)$ on the same hull. We also store the index of the inner extreme edge in this array. These arrays can be computed in total $O(n)$ time by traversing the upper and the lower hull from right to left.

The precomputation takes $O(n)$ time in total. Given an extreme column pair (p, e) , let l_e be the left endpoint of e . We can use precomputation (ii) to find in $O(1)$ time the closest extreme vertex p' to the left of p , since it is either $\lambda(p)$ or $\lambda(l_e)$, as well as the distance w' , which is either $\delta(p)$ or $\delta(l_e)$. To test whether we can canonically draw the polygon between (p', e') and (p, e) , we make use of the fact that there is no outer extreme edge between them. Hence, we only have to test whether a pair of opposite double stairs intersects. To this end, we observe that a pair of double stairs can only intersect if the inner extreme edge of the lower hull lies (partially) above the upper hull or the inner extreme edge of the upper hull lies (partially) below the lower hull. With the array precomputed in step (iii), we can find the edge opposite of the inner extreme edges, and by precomputation (i), each point (and thus each edge) knows its y -coordinate, which we only have to compare to find out whether an intersection exists. Hence, we can compute each table entry in $O(1)$ times after a precomputation step that takes $O(n)$ time.

Since we call the dynamic program $O(n)$ times—once for each candidate for the height of the bounding box—the algorithm takes $O(n^3)$ time in total. Following Lemma 3, this proves the theorem. \square

For the area minimization, we make two key observations. First, since the polygon is x -monotone, each grid column (properly) intersects either no or exactly two horizontal edges: one edge from the upper hull and one edge from the lower hull. Second, a pair of horizontal edges share at most one column; otherwise, the polygon could be drawn with less area by shortening both edges. With the same argument as for the bounding box, the height of any minimum-area polygon is at most n .

We use a dynamic program to solve the problem. To this end, we fill a three-dimensional table T as follows. Let e be a horizontal edge on the upper hull, let f be a horizontal edge of the lower hull, and let $1 \leq h \leq n$. Then, the entry $T[e, f, h]$ specifies the minimum area required to draw the part of the polygon to the left of (and including) the unique common column of e and f under the condition that e and f share a column and have vertical distance h .

Let e_1, \dots, e_k be the horizontal edges on the upper hull from left to right and let f_1, \dots, f_m be the horizontal edges on the lower hull from left to right. We initialize the table with $T[e_1, f_1, h] = h$ for each $1 \leq h \leq n$. To compute any other entry $T[e_i, f_j, h']$, we need to find the correct entry from the column left of the column shared by e_i and f_j . There are three possibilities: this column either intersects e_{i-1} and f_{j-1} , it intersects e_i and f_{j-1} , or it intersects e_{i-1} and f_j . For each of these possibilities, we check which height can be realized if e_i and f_j have vertical distance h' and search for the entry of minimum value. We set

$$T[e_i, f_j, h'] = \min_{h'' \text{ valid}} \{T[e_{i-1}, f_{j-1}, h''], T[e_i, f_{j-1}, h''], T[e_{i-1}, f_j, h'']\} + h'.$$

Finally, we can find the optimum solution by finding $\min_{1 \leq h \leq n} \{T[e_k, f_m, h]\}$. Since the table has $O(n^3)$ entries each of which we can compute in $O(n)$ time, the algorithm runs in $O(n^4)$ time. This proves the following theorem.

Theorem 6. *Given an x -monotone angle sequence S of length n , we can find a minimum-area polygon that realizes S in $O(n^4)$ time.*

4 The Monotone Case: Minimum Perimeter

In this section, we show how to compute a polygon of minimum perimeter for an xy -monotone or x -monotone angle sequence S of length n .

Let P be an x -monotone polygon realizing S . Let e_L be the leftmost vertical edge and let e_R be the rightmost vertical edge of P . Recall that P consists of two x -monotone chains; an upper chain T and a lower chain B connected by e_L and e_R . Without loss of generality, we assume for the number of reflex vertices of T and B that $r(T) \geq r(B)$.

An x -monotone polygon is *perimeter-canonical* if (P1) every vertical edge except e_R and e_L has unit length, and (P2) every horizontal edge of T has unit length. We show that it suffices to find a perimeter-canonical polygon of minimum perimeter.

Lemma 4. *Any x -monotone polygon can be transformed into a perimeter-canonical x -monotone polygon without increasing its perimeter.*

Proof. We transform any minimum-perimeter polygon into a perimeter-canonical form without sacrificing its perimeter in two steps as follows. First, we shorten every *long* vertical edge $e \in T \cup B$ with $|e| > 1$ so that $|e| = 1$. This is always possible: For any long vertical edge $e \in T \cup B$, say $e \in T$, if its end vertices have turns RL in counterclockwise order, then we proceed as in Figure 12(a). We move the subchain $T(e_R, e) \subseteq T$ from e_R to e upward by $|e| - 1$ by shortening e and simultaneously by stretching e_R . This guarantees that $|e|$ decreases, instead $|e_R|$ increases by the same amount of $|e| - 1$, so the perimeter remains the same. We can also shorten any long vertical edge whose end vertices have turns LR in a symmetric way.

Second, we shorten every long horizontal edge $e \in T$ with $|e| > 1$ so that its length becomes one. Suppose that e is the rightmost long horizontal edge e in T . Since $r(T) \geq r(B)$, there must be a long horizontal edge e' in B . We shorten both e and e' by one unit, and move two subchains $T(e_R, e)$ and $B(e', e_R)$ together with e_R one unit left. This move may cause two vertical edges, $f \in T$ and $f' \in B$, to intersect; see Figure 12(b). Note that exactly one of both vertical edges did not move, say f' , as otherwise there would be no intersection between them. This means f' is to the left of e' , i.e., $f' \in B \setminus B(e', e_R)$. We also know that the x -distance between f and f' prior to the move was one, otherwise they would not intersect. Since f and f' are of unit length, the lower end vertex of f has the same y -coordinate as the upper end vertex of f' . To avoid the intersection, we first move the whole upper chain T one unit upward by stretching e_R and e_L each by one unit, as in Figure 12(c). Then we can move $T(e_R, e)$, $B(e', e_R)$, and e_R without causing any intersection. We lose two units by shortening e and e' , and gain two units by stretching e_R and e_L , so the total perimeter is unchanged. We repeat this until $|e| = 1$. \square

Suppose that P is a minimum-perimeter canonical polygon that realizes S with $r(T) \geq r(B)$, and $\text{peri}(P)$ denotes its perimeter. By conditions (P1)–(P2), every edge in T is of unit length, so the length of T is $2r(T) + 1$. This implies the width of B should be $r(T) + 1$. By condition (P1), the length of the vertical edges in B is $r(B)$, so the total length of B is $r(T) + r(B) + 1$. Thus we can observe the following property.

Lemma 5. *Given an x -monotone angle sequence S , there is a canonical minimum-perimeter polygon P realizing S with $r(T) \geq r(B)$ such that $\text{peri}(P) = 3r(T) + r(B) + 2 + |e_L| + |e_R|$.*

The first three terms of $\text{peri}(P)$ in Lemma 5 are constant, so we need to minimize the sum of the last two terms, $|e_L|$ and $|e_R|$, to get a minimum perimeter. However, once one of them is fixed, the other is automatically determined by the fact that all vertical edges in T and B are unit segments. Even more, minimizing one of them is equivalent to

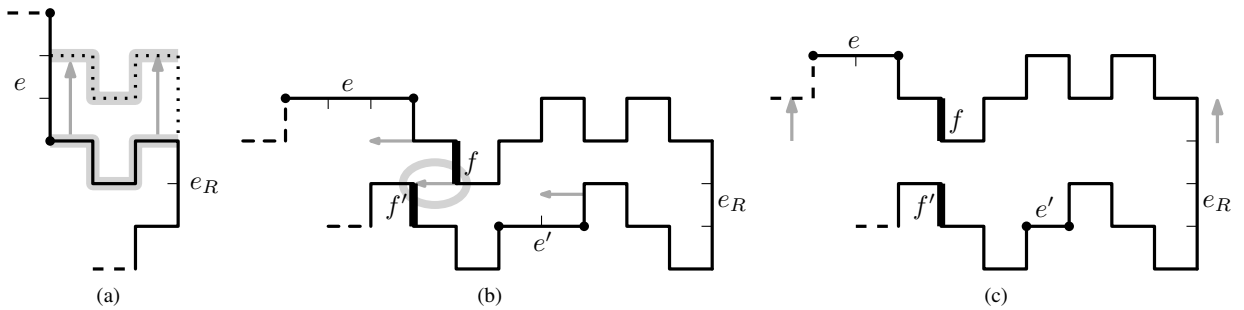


Figure 12: Transforming P into a canonical form.

minimizing their sum, consequently minimizing the perimeter. We call the length of the left vertical extreme edge of a polygon the *height* of the polygon.

4.1 The xy -Monotone Case

Let P be a minimum-perimeter canonical xy -monotone polygon that realizes an xy -monotone angle sequence S of length n . As before, we assume that $r(T) \geq r(B)$. When $n = 4$, i.e., the number r of reflex vertices is 0, a unit square P achieves the minimum perimeter, so we assume here that $r > 0$. Recall that the boundary of P consists of four stairs, TR , TL , BL , and BR . Let (r_1, r_2, r_3, r_4) be a quadruple of the numbers of reflex vertices of TR , TL , BL , and BR , respectively. Then $r = r_1 + r_2 + r_3 + r_4$, where $r_i \geq 0$ for each i . Again, we define L as the chain consisting of TL , e_L and BL and R as the chain consisting of BR , e_R and TR . In P , let $w(T)$ and $w(B)$ denote the widths of T and B , respectively, and $h(L)$ and $h(R)$ the heights of L and R , respectively. Hence, the perimeter of P is $\text{peri}(P) = w(T) + w(B) + h(L) + h(R)$.

Note that $w(T) = w(B)$ and, by condition (P2), $w(T) = r_1 + 1 + r_2$. Thus $w(T) + w(B) = 2(r_1 + r_2) + 2$. Similarly, $h(L) = h(R)$, and, by condition (P1), $h(L) = r_2 + |e_L| + r_3$ and $h(R) = r_4 + |e_R| + r_1$. Thus, if $|e_L| = 1$, then $h(L) + h(R) = 2(r_2 + r_3) + 2$, and, if $|e_R| = 1$, then $h(L) + h(R) = 2(r_1 + r_4) + 2$. Further observe that $|e_L| = 1$ implies $r_2 + r_3 \geq r_1 + r_4$, and that $|e_R| = 1$ implies $r_2 + r_3 \leq r_1 + r_4$. Hence, if $|e_L| = 1$ or $|e_R| = 1$, then $h(L) + h(R) = r + |r_2 + r_3 - r_1 - r_4|$ and then

$$\text{peri}(P) = 3(r_1 + r_2) + (r_3 + r_4) + |r_2 + r_3 - r_1 - r_4| + 4. \quad (1)$$

Now, consider the remaining case when $|e_L| \geq 2$ and $|e_R| \geq 2$. We will observe that this case can occur only if (r_1, r_2, r_3, r_4) is $(r_1, 0, r_1, 0)$ or $(0, r_2, 0, r_2)$. We will also observe that then $|e_L| = |e_R| = 2$. Hence, we get that $\text{peri}(P) = 2r_1 + 6$ for case $(r_1, 0, r_1, 0)$, and $\text{peri}(P) = 2r_2 + 6$ for case $(0, r_2, 0, r_2)$. For all other cases, equation 1 holds.

To make these observations, we first apply the same contraction step as depicted in Figure 10(b) of Lemma 3. That is, we contract all horizontal segments of BL to length 1 by moving all their right endpoints as far as possible to the left, and we contract all horizontal segments of BR to length 1 by moving all their left endpoints as far as possible to the right. By this, all edges of B except the bottom extreme edge have length 1, and the perimeter does not change. Next, note that T and B have vertical distance 1 to each other. Otherwise we could move B at least one unit to the top by simultaneously shrinking e_L and e_B , and thus shrinking the perimeter of P , a contradiction to the minimality of $\text{peri}(P)$. As T consists only of unit segments (conditions (P1)–(P2)), there is a vertex p in T having distance 1 to B .

First assume that p belongs to TR . We choose the rightmost such p . If p were a convex vertex, then it would be the top endpoint of e_R , and, hence, we would have $|e_R| = 1$; a contradiction to $|e_R| \geq 2$. Thus, p is a reflex vertex and therefore an left endpoint of a horizontal edge pp' . Hence, the right endpoint p' of pp' is convex. Let e be the edge in B below pp' , that is, the edge that crosses the same grid column as pp' . Observe that the distance between pp' and e is at least 2. If it were 1, then the vertical edge $p'p''$ incident to p' would connect to e (recall that p' is convex). Hence, pp' and e would be incident to $e_R = p'p''$, and again we would have $|e_R| = 1$; a contradiction. Thus, the distance between p and e is at least 2. Let q be the point of B directly one unit below p . Then e lies at least one unit below q . Hence, q has to connect to e via a vertical edge, and, consequently, q has to be a reflex vertex and belong to BL . By condition (P1), the vertical edge connecting q and e has length 1, hence, the distance between pp' and e is exactly 2. But now, either the bottom endpoint p'' of $p'p''$ has distance 1 to B , or p'' lies on B , that is, $p'p'' = e_R$. The former case contradicts our assumption that p is the rightmost vertex of T having distance 1 to B . Thus, the latter case holds and pp' and e are incident to e_R . Hence, $|e_R| = 2$, e is the bottom extreme edge and has length $|e| = 1$, and BR is empty, that is, $r_4 = 0$. Thus, all horizontal edges in B have unit length. This property allows us to use the same argument as above to show that $r_2 = 0$ and $|e_L| = 2$. Given $r_1 + 1 = w(T) = w(B) = r_3 + 1$, we get $r_1 = r_3$.

Finally, assume that p belongs to TL . Then we can show in a similar way as above that we are in case $(0, r_2, 0, r_2)$, and, again, $|e_L| = |e_R| = 2$. Thus our observation follows.

Theorem 7. *Given an xy -monotone angle sequence S of length n , we can find a polygon P that realizes S and minimizes its perimeter in $O(n)$ time. Furthermore, if the lengths of the stair sequences (r_1, r_2, r_3, r_4) are given as*

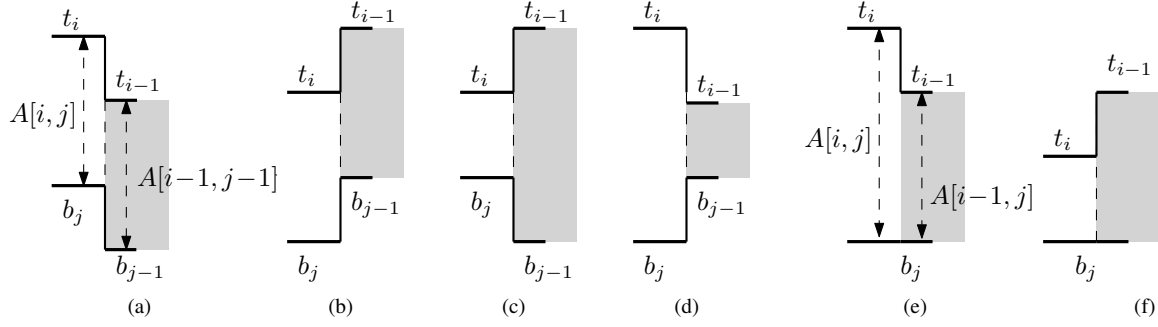


Figure 13: Six situations when t_i and b_j are considered to fill $A[i, j]$.

above, then $\text{peri}(P)$ can be expressed as:

$$\text{peri}(P) = \begin{cases} 4r_1 + 6 & \text{if } (r_1, 0, r_1, 0), \\ 4r_2 + 6 & \text{if } (0, r_2, 0, r_2), \\ 3(r_1 + r_2) + (r_3 + r_4) + |r_3 - (r_1 - r_2 + r_4)| + 4 & \text{otherwise.} \end{cases}$$

4.2 The x -Monotone Case

A minimum height polygon P that realizes S can be computed in $O(n^2)$ time using dynamic programming. Recall that a perimeter-canonical polygon of minimum height is a polygon of minimum perimeter.

From right to left, let $t_1, t_2, \dots, t_{r(T)}$ be the horizontal edges in T and $b_1, b_2, \dots, b_{r(B)}$ be the horizontal edges in B . Recall that $r(T) \geq r(B)$. For $i \geq j \geq 1$, let $A[i, j]$ be the minimum height of the subpolygon formed with the first i horizontal edges from T and the first j horizontal edges from B . Note that the leftmost vertical edge of the subpolygon whose minimum height is stored in $A[i, j]$ joins the left endpoints of t_i and b_j . To compute $A[i, j]$, we attach edges t_i and b_j to the upper and lower chains of the subpolygon constructed so far. Since t_i has unit length, either t_i and b_j are attached to the subpolygon with height of $A[i-1, j-1]$ or just t_i is attached to the subpolygon with height of $A[i-1, j]$. As in Figure 13, there are four cases (a)–(d) for the first attachment and two cases (e)–(f) for the second attachment, according to the turns formed at the attachments.

Let u and v be the left end vertex of t_{i-1} and the right end vertex of t_i , respectively. Let u' and v' be the right end vertex of b_j and the left end vertex of b_{j-1} , respectively. Notice that both vertical edges (u, v) and (u', v') have unit length. As an example, let us explain how to calculate $A[i, j]$ when $uv = \text{LR}$ and $u'v' = \text{LR}$, which corresponds to Figures 13(b) and (f). We set $A[i, j]$ to the minimum height of the two possible attachments (b) and (f). Consider the height for (b). If $A[i-1, j-1] > 1$, then t_i and b_j are attached to the subpolygon as illustrated in Figure 13(b). Since edges (u, v) and (u', v') have unit length, $A[i, j] = A[i-1, j-1]$. In the other case, if $A[i-1, j-1] = 1$, then we can move the upper chain of the subpolygon one unit upward without intersection so that t_i and b_j are safely attached to the subpolygon with $A[i, j] = 2$. Note that this is the smallest possible value for $A[i, j]$ given $uv = \text{LR}$ and $u'v' = \text{LR}$. Thus $A[i, j] = \max(A[i-1, j-1], 2)$. The height for (f) should be at least 1, so it is expressed as $\max(A[i-1, j] - 1, 1)$. Therefore,

$$A[i, j] = \min(\max(A[i-1, j-1], 2), \max(A[i-1, j] - 1, 1)).$$

For the other turns at uv and $u'v'$, we can similarly define the equations as follows:

$$A[i, j] = \begin{cases} \text{undefined} & \text{if } i = 0, j = 0 \text{ or } i < j \\ 1 & \text{if } i = 1, j = 1 \\ A[i - 1, j] + 1 & \text{if } uv = \text{RL}, j = 1 \\ \max(A[i - 1, j] - 1, 1) & \text{if } uv = \text{LR}, j = 1 \\ \min(\max(A[i - 1, j - 1], 2), A[i - 1, j] + 1) & \text{if } uv = \text{RL}, u'v' = \text{RL} \\ \min(\max(A[i - 1, j - 1], 2), \max(A[i - 1, j] - 1, 1)) & \text{if } uv = \text{LR}, u'v' = \text{LR} \\ \min(A[i - 1, j - 1] + 2, A[i - 1, j] + 1) & \text{if } uv = \text{RL}, u'v' = \text{LR} \\ \min(\max(A[i - 1, j - 1] - 2, 1), \max(A[i - 1, j] - 1, 1)) & \text{if } uv = \text{LR}, u'v' = \text{RL} \end{cases}$$

Evaluating each entry takes constant time, so the total time to fill A is $O(n^2)$. Using A , a minimum-perimeter polygon can be reconstructed within the same time bound.

Theorem 8. *Given an x -monotone angle sequence S of length n , we can find a polygon P that realizes S and minimizes its perimeter in $O(n^2)$ time.*

References

- [1] S. W. Bae, Y. Okamoto, and C. Shin. Area bounds of rectilinear polygons realized by angle sequences. In K. Chao, T. Hsu, and D. Lee, editors, *Proc. 23rd Int. Symp. Algorithms Comput. (ISAAC'12)*, volume 7676 of *LNCS*, pages 629–638. Springer, 2012.
- [2] T. C. Biedl, S. Durocher, and J. Snoeyink. Reconstructing polygons from scanner data. *Theor. Comput. Sci.*, 412(32):4161–4172, 2011.
- [3] D. Z. Chen and H. Wang. An improved algorithm for reconstructing a simple polygon from its visibility angles. *Comput. Geom.*, 45(5-6):254–257, 2012.
- [4] J. C. Culbertson and G. J. E. Rawlins. Turtlegons: Generating simple polygons from sequences of angles. In *Proc. 1st Ann. ACM Symp. Comp. Geom. (SoCG'85)*, pages 305–310, 1985.
- [5] Y. Disser, M. Mihalák, and P. Widmayer. A polygon is determined by its angles. *Comput. Geom.*, 44(8):418–426, 2011.
- [6] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman & Co., NY, US, 1979.
- [7] R. I. Hartley. Drawing polygons given angle sequences. *Inform. Process. Lett.*, 31(1):31–33, 1989.
- [8] M. Patrignani. On the complexity of orthogonal compaction. *Comput. Geom.*, 19(1):47–67, 2001.
- [9] J.-R. Sack. *Rectilinear Computational Geometry*. PhD thesis, School of Computer Science, McGill University, 1984.
- [10] R. Tamassia. On embedding a graph in the grid with the minimum number of bends. *SIAM J. Comput.*, 16(3):421–444, 1987.
- [11] G. Vijayan and A. Wigderson. Rectilinear graphs and their embeddings. *SIAM J. Comput.*, 14(2):355–372, 1985.